Mathematics Extension 1

General Instructions
- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84
- Attempt Questions 1–7
- All questions are of equal value
Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) The polynomial $x^3$ is divided by $x + 3$. Calculate the remainder.  
   
   Marks: 2

(b) Differentiate $\cos^{-1}(3x)$ with respect to $x$.  
   
   Marks: 2

(c) Evaluate $\int_{-1}^{1} \frac{1}{\sqrt{4 - x^2}} dx$.  
   
   Marks: 2

(d) Find an expression for the coefficient of $x^8y^4$ in the expansion of $(2x + 3y)^{12}$.  
   
   Marks: 2

(e) Evaluate $\int_{0}^{\pi} \cos\theta \sin^2\theta d\theta$.  
   
   Marks: 2

(f) Let $f(x) = \log_e[(x - 3)(5 - x)]$.  
   
   What is the domain of $f(x)$?  
   
   Marks: 2
Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Use the substitution \( u = \log_e x \) to evaluate
\[
\int_e^{e^2} \frac{1}{x (\log_e x)^2} \, dx.
\]

(b) A particle moves on the \( x \)-axis with velocity \( v \). The particle is initially at rest at \( x = 1 \). Its acceleration is given by \( \ddot{x} = x + 4 \).

Using the fact that \( \ddot{x} = \frac{d}{dx} \left( \frac{1}{2} v^2 \right) \), find the speed of the particle at \( x = 2 \).

(c) The polynomial \( p(x) \) is given by \( p(x) = ax^3 + 16x^2 + cx - 120 \), where \( a \) and \( c \) are constants.

The three zeros of \( p(x) \) are \(-2, 3 \) and \( \alpha \).

Find the value of \( \alpha \).

(d) The function \( f(x) = \tan x - \log_e x \) has a zero near \( x = 4 \).

Use one application of Newton’s method to obtain another approximation to this zero. Give your answer correct to two decimal places.
Question 3 (12 marks) Use a SEPARATE writing booklet.

(a)   (i) Sketch the graph of \( y = |2x - 1| \).

(ii) Hence, or otherwise, solve \( |2x - 1| \leq |x - 3| \).

(b) Use mathematical induction to prove that, for integers \( n \geq 1 \),

\[
1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n + 2) = \frac{n}{6}(n + 1)(2n + 7).
\]

(c) A race car is travelling on the \( x \)-axis from \( P \) to \( Q \) at a constant velocity, \( v \).
A spectator is at \( A \) which is directly opposite \( O \), and \( OA = \ell \) metres. When the car is at \( C \), its displacement from \( O \) is \( x \) metres and \( \angle OAC = \theta \), with \( \frac{\pi}{2} < \theta < \frac{\pi}{2} \).

(i) Show that \( \frac{d\theta}{dt} = \frac{v\ell}{\ell^2 + x^2} \).

(ii) Let \( m \) be the maximum value of \( \frac{d\theta}{dt} \).
Find the value of \( m \) in terms of \( v \) and \( \ell \).

(iii) There are two values of \( \theta \) for which \( \frac{d\theta}{dt} = \frac{m}{4} \).
Find these two values of \( \theta \).
Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) A turkey is taken from the refrigerator. Its temperature is 5°C when it is placed in an oven preheated to 190°C.

Its temperature, $T^\circ C$, after $t$ hours in the oven satisfies the equation

$$\frac{dT}{dt} = -k(T - 190).$$

(i) Show that $T = 190 - 185e^{-kt}$ satisfies both this equation and the initial condition.

(ii) The turkey is placed into the oven at 9 am. At 10 am the turkey reaches a temperature of 29°C. The turkey will be cooked when it reaches a temperature of 80°C.

At what time (to the nearest minute) will it be cooked?

(b) Barbara and John and six other people go through a doorway one at a time.

(i) In how many ways can the eight people go through the doorway if John goes through the doorway after Barbara with no-one in between?

(ii) Find the number of ways in which the eight people can go through the doorway if John goes through the doorway after Barbara.

Question 4 continues on page 7
The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The tangents to the parabola at $P$ and $Q$ intersect at $T$. The chord $QO$ produced meets $PT$ at $K$, and $\angle PKQ$ is a right angle.

(i) Find the gradient of $QO$, and hence show that $pq = -2$.  

(ii) The chord $PO$ produced meets $QT$ at $L$. Show that $\angle PLQ$ is a right angle. 

(iii) Let $M$ be the midpoint of the chord $PQ$. By considering the quadrilateral $PQLK$, or otherwise, show that $MK = ML$.

End of Question 4
Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Let \( f(x) = x - \frac{1}{2}x^2 \) for \( x \leq 1 \). This function has an inverse, \( f^{-1}(x) \).

(i) Sketch the graphs of \( y = f(x) \) and \( y = f^{-1}(x) \) on the same set of axes. (Use the same scale on both axes.)

(ii) Find an expression for \( f^{-1}(x) \).

(iii) Evaluate \( f^{-1}\left(\frac{3}{8}\right) \).

(b) A particle is moving in simple harmonic motion in a straight line. Its maximum speed is 2 m s\(^{-1}\) and its maximum acceleration is 6 m s\(^{-2}\).

Find the amplitude and the period of the motion.

(c) Two circles \( C_1 \) and \( C_2 \) intersect at \( P \) and \( Q \) as shown in the diagram. The tangent \( TP \) to \( C_2 \) at \( P \) meets \( C_1 \) at \( K \). The line \( KQ \) meets \( C_2 \) at \( M \). The line \( MP \) meets \( C_1 \) at \( L \).

Copy or trace the diagram into your writing booklet.

Prove that \( \triangle PKL \) is isosceles.
Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) From a point $A$ due south of a tower, the angle of elevation of the top of the tower $T$, is $23^\circ$. From another point $B$, on a bearing of $120^\circ$ from the tower, the angle of elevation of $T$ is $32^\circ$. The distance $AB$ is 200 metres.

(i) Copy or trace the diagram into your writing booklet, adding the given information to your diagram.

(ii) Hence find the height of the tower.

(b) It can be shown that $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$ for all values of $\theta$. (Do NOT prove this.)

Use this result to solve $\sin 3\theta + \sin 2\theta = \sin \theta$ for $0 \leq \theta \leq 2\pi$.

(c) Let $p$ and $q$ be positive integers with $p \leq q$.

(i) Use the binomial theorem to expand $(1 + x)^{p+q}$, and hence write down the term of $\frac{(1 + x)^{p+q}}{x^q}$ which is independent of $x$.

(ii) Given that $\frac{(1 + x)^{p+q}}{x^q} = (1 + x)^p \left( 1 + \frac{1}{x} \right)^q$, apply the binomial theorem and the result of part (i) to find a simpler expression for

$$1 + \binom{p}{1} \binom{q}{1} + \binom{p}{2} \binom{q}{2} + \ldots + \binom{p}{p} \binom{q}{p}.$$
A projectile is fired from $O$ with velocity $V$ at an angle of inclination $\theta$ across level ground. The projectile passes through the points $L$ and $M$, which are both $h$ metres above the ground, at times $t_1$ and $t_2$ respectively. The projectile returns to the ground at $N$.

The equations of motion of the projectile are

$$x = Vt \cos \theta \text{ and } y = Vt \sin \theta - \frac{1}{2} gt^2.$$ (Do NOT prove this.)

(a) Show that $t_1 + t_2 = \frac{2V}{g} \sin \theta$ AND $t_1 t_2 = \frac{2h}{g}$. 

Question 7 continues on page 11
Question 7 (continued)

Let \( \angle LON = \alpha \) and \( \angle LNO = \beta \). It can be shown that

\[
\tan \alpha = \frac{h}{V_{l_1} \cos \theta} \quad \text{and} \quad \tan \beta = \frac{h}{V_{l_2} \cos \theta}. \quad \text{(Do NOT prove this.)}
\]

(b) Show that \( \tan \alpha + \tan \beta = \tan \theta \). 2

(c) Show that \( \tan \alpha \tan \beta = \frac{gh}{2V^2 \cos^2 \theta} \). 1

Let \( ON = r \) and \( LM = w \).

(d) Show that \( r = h(\cot \alpha + \cot \beta) \) and \( w = h(\cot \beta - \cot \alpha) \). 2

Let the gradient of the parabola at \( L \) be \( \tan \phi \).

(e) Show that \( \tan \phi = \tan \alpha - \tan \beta \). 3

(f) Show that \( \frac{w}{\tan \phi} = \frac{r}{\tan \theta} \). 2

End of paper
STANDARD INTEGRALS

\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0 \]

\[ \int \frac{1}{x} \, dx = \ln x, \quad x > 0 \]

\[ \int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0 \]

\[ \int \cos(ax) \, dx = \frac{1}{a} \sin(ax), \quad a \neq 0 \]

\[ \int \sin(ax) \, dx = -\frac{1}{a} \cos(ax), \quad a \neq 0 \]

\[ \int \sec^2(ax) \, dx = \frac{1}{a} \tan(ax), \quad a \neq 0 \]

\[ \int \sec(ax) \tan(ax) \, dx = \frac{1}{a} \sec(ax), \quad a \neq 0 \]

\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0 \]

\[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a \]

\[ \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0 \]

\[ \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left( x + \sqrt{x^2 + a^2} \right) \]

NOTE:  \( \ln x = \log_e x, \quad x > 0 \)